JEE ADVANCED

Single Correct Answer Type

1. The least value of the expression $2\log_{10} x - \log_x(0.01)$, for $x > 1$, is
   a. 10 **b.** 2 **c.** -0.01 **d.** none of these

2. The third term of a geometric progression is 4. The product of the first five terms is
   a. 4^3 **b.** 4^5 **c.** 4^4 **d.** none of these
   (IIT-JEE 1982)

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   a. 10 b. 2 c. -0.01 d. none of these

2. The third term of a geometric progression is 4. The product of the first five terms is
   a. 4^3 b. 4^5 c. 4^4 d. none of these
   (IIT-JEE 1982)

3. The rational number which equals the number $2.\overline{357}$ with recurring decimal is
   a. $\frac{2355}{1001}$ b. $\frac{2379}{997}$ c. $\frac{2355}{999}$ d. none of these
   (IIT-JEE 1983)

4. If a, b, and c are in G.P., then the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root if $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in
   a. A.P. b. G.P. c. H. P. d. none of these
   (IIT-JEE 1985)

5. Sum of the first n terms of the series $\frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \frac{15}{16} + \cdots$ is equal to
   a. $2^n - n - 1$ b. $1 - 2^{-n}$ c. $n + 2^{-n} - 1$ d. $2^n + 1$
   (IIT-JEE 1988)

6. Find the sum $(x+2)^{n-1} + (x+2)^{n-2}(x+1) + (x+2)^{n-3}(x+1)^2 + \cdots + (x+1)^{n-1}$
   a. $(x+2)^{n-2} - (x+1)^n$ b. $(x+2)^{n-1} - (x+1)^{n-1}$
   c. $(x+2)^n - (x+1)^n$ d. none of these
   (IIT-JEE 1990)

7. If x, y, and z are pth, qth, and rth terms, respectively, of an A.P. and also of a G.P., then $x^{y-z}\, y^{z-x}\, z^{x-y}$ is equal to
   a. xyz b. 0 c. 1 d. none of these
   (IIT-JEE 1991)

8. The product of n positive numbers is unity. Then their sum is
   a. a positive integer b. divisible by n
   c. equals to $n + 1/n$ d. never less than n
   (IIT-JEE 1991)

9. If $\ln(a + c)$, $\ln(a - c)$, and $\ln(a - 2b + c)$ are in A.P., then
   a. a, b, c are in A.P. b. a^2, b^2, c^2 are in A.P.
   c. a, b, c are in G.P. d. a, b, c are in H.P.
   (IIT-JEE 1994)

10. Let T_r be the rth term of an A.P., for $r = 1, 2, 3, \ldots$. If for some positive integers m, n, we have $T_m = \frac{1}{n}$ and $T_n = \frac{1}{m}$, then T_{mn} equals
    a. $\frac{1}{mn}$ b. $\frac{1}{m} + \frac{1}{n}$ c. 1 d. 0
    (IIT-JEE 1998)

11. If $x > 1, y > 1$, and $z > 1$ are in G.P., then $\frac{1}{1+\ln x}, \frac{1}{1+\ln y}$, and $\frac{1}{1+\ln z}$ are in
    a. A.P. b. H.P. c. G.P. d. none of these
    (IIT-JEE 1998)

12. Let $a_1, a_2, \ldots, a_{10}$ be in A.P. and $h_1, h_2, \ldots, h_{10}$ be in H.P. If $a_1 = h_1 = 2$ and $a_{10} = h_{10} = 3$, then a_4h_7 is
    a. 2 b. 3 c. 5 d. 6
    (IIT-JEE 1999)

13. The harmonic mean of the roots of the equation $(5+\sqrt{2})x^2 - (4+\sqrt{5})x + 8 + 2\sqrt{5} = 0$ is
    a. 2 b. 4 c. 6 d. 8
    (IIT-JEE 1999)

14. If a, b, c are different positive real numbers such that $b + c - a$, $c + a - b$, and $a + b - c$ are positive, then $(b + c - a)(c + a - b)(a + b - c) - abc$ is
    a. positive b. negative
    c. non-positive d. non-negative
    (IIT-JEE 1999)

15. If a, b, c, d are positive real numbers such that $a + b + c + d = 2$, then $M = (a + b)(c + d)$ satisfies the relation
    a. $0 \le M \le 1$ b. $1 \le M \le 2$
    c. $2 \le M \le 3$ d. $3 \le M \le 4$
    (IIT-JEE 2000)

16. Let the positive numbers a, b, c, and d be in A.P. Then abc, abd, acd, and bcd are
    a. not in A.P./G.P./H.P. b. in A.P.
    c. in G.P. d. in H.P. **(IIT-JEE 2001)**

17. Consider an infinite geometric series with first term a and common ratio r. If its sum is 4 and the second term is 3/4, then
    a. $a = \frac{4}{7}, r = \frac{3}{7}$ b. $a = 2, r = \frac{3}{8}$
    c. $a = \frac{3}{2}, r = \frac{1}{2}$ d. $a = 3, r = \frac{1}{4}$
    (IIT-JEE 2001)

18. Let α and β be the roots of $x^2 - x + p = 0$ and γ and δ be the root of $x^2 - 4x + q = 0$. If α, β, and γ, δ are in G.P., then the integral values of p and q, respectively, are
    a. $-2, -32$ b. $-2, 3$ c. $-6, 3$ d. $-6, -32$
    (IIT-JEE 2001)

19. If the sum of the first $2n$ terms of the A.P. 2, 5, 8, …, is equal to the sum of the first n terms of A.P. 57, 59, 61, …, then n equals
    a. 10 b. 12 c. 11 d. 13
    (IIT-JEE 2001)

20. Suppose a, b, and c are in A.P. and a^2, b^2, and c^2 are in G.P. If $a < b < c$ and $a + b + c = \frac{3}{2}$, then the value of a is
    a. $\frac{1}{2\sqrt{2}}$ b. $\frac{1}{2\sqrt{3}}$ c. $\frac{1}{2} - \frac{1}{\sqrt{3}}$ d. $\frac{1}{2} - \frac{1}{\sqrt{2}}$
    (IIT-JEE 2002)

21. If $a_1, a_2, \ldots, a_n$ are positive real numbers whose product is a fixed number c, then the minimum value of $a_1 + a_2 + \cdots + a_{n-1} + 2a_n$ is
    a. $n(2c)^{1/n}$ b. $(n+1)c^{1/n}$
    c. $2nc^{1/n}$ d. $(n+1)(2c)^{1/n}$
    (IIT-JEE 2002)

22. An infinite G.P. has first term as a and sum 5, then

23. If $\alpha \in \left(0, \frac{\pi}{2}\right)$, then $\sqrt{x^2+x}+\frac{\tan^2\alpha}{\sqrt{x^2+x}}$ is always greater than or equal to

a. $2\tan\alpha$ **b.** 1

c. 2 **d.** $\sec^2\alpha$

(IIT-JEE 2004)

24. If the sum of first n terms of an A.P. is cn^2, then the sum of squares of these n terms is

a. $\frac{n(4n^2-1)c^2}{6}$ **b.** $\frac{n(4n^2+1)c^2}{3}$

c. $\frac{n(4n^2-1)c^2}{3}$ **d.** $\frac{n(4n^2+1)c^2}{6}$

(IIT-JEE 2009)

25. Let $a_1, a_2, a_3, \ldots$ be in harmonic progression with $a_1 = 5$ and $a_{20} = 25$. The least positive integer n for which $a_n < 0$

a. 22 **b.** 23 **c.** 24 **d.** 25

(IIT-JEE 2012)

Multiple Correct Answers Type

1. If the first and the $(2n-1)$st terms of an A.P., a G.P., and a H.P. are equal and their nth terms are a, b, and c, respectively, then

a. $a = b = c$ **b.** $a \ge b \ge c$ **c.** $a + b = b$ **d.** $ac - b^2 = 0$

(IIT-JEE 1988)

2. For $0 < \phi < \pi/2$, if

$$x = \sum_{n=0}^{\infty}\cos^{2n}\phi,\ y = \sum_{n=0}^{\infty}\sin^{2n}\phi, \text{ and } z = \sum_{n=0}^{\infty}\cos^{2n}\phi\sin^{2n}\phi,$$

then

a. $xyz = xz + y$ **b.** $xyz = xy + z$

c. $xyz = x + y + z$ **d.** $xyz = yz + x$

(IIT-JEE 1993)

3. For a positive integer n, let $a(n) = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\cdots+\frac{1}{(2^n)-1}$. Then,

a. $a(100) \le 100$ **b.** $a(100) > 100$

c. $a(200) \le 100$ **d.** $a(200) > 100$

(IIT-JEE 1999)

4. A straight line through the vertex P of a triangle PQR intersects the side QR at the point S and the circumcircle of the triangle PQR at the point T. If S is not the center of the circumcircle, then

a. $\frac{1}{PS}+\frac{1}{ST} < \frac{2}{\sqrt{QS \times SR}}$ **b.** $\frac{1}{PS}+\frac{1}{ST} > \frac{2}{\sqrt{QS \times SR}}$

c. $\frac{1}{PS}+\frac{1}{ST} < \frac{4}{QR}$ **d.** $\frac{1}{PS}+\frac{1}{ST} > \frac{4}{QR}$

(IIT-JEE 2008)

5. Let $S_n = \sum_{k=1}^{4n}(-1)^{\frac{k(k+1)}{2}}k^2$. Then S_n can take value(s)

a. 1056 **b.** 1088 **c.** 1120 **d.** 1332

(JEE Advanced 2013)

Linked Comprehension Type

For Problems 1–3

Let V_r denote the sum of first r terms of an arithmetic progression (A.P.) whose first term is r and the common difference is $2r-1$. Let $T_r = V_{r+1} - V_r - 2$ and $Q_r = T_{r+1} - T_r$ for $r = 1, 2, \ldots$. **(IIT-JEE 2007)**

1. The sum $V_1 + V_2 + \cdots + V_n$ is

a. $\frac{1}{12}n(n+1)(3n^2-n+1)$ **b.** $\frac{1}{12}n(n+1)(3n^2+n+2)$

c. $\frac{1}{2}n(2n^2-n+1)$ **d.** $\frac{1}{3}(2n^3-2n+3)$

2. T_r is always

a. an odd number **b.** an even number

c. a prime number **d.** a composite number

3. Which one of the following is a correct statement?

a. $Q_1, Q_2, Q_3, \ldots$ are in A. P. with common difference 5

b. $Q_1, Q_2, Q_3, \ldots$ are in A. P. with common difference 6

c. $Q_1, Q_2, Q_3, \ldots$ are in A. P. with common difference 11

d. $Q_1 = Q_2 = Q_3, = \cdots$

For Problems 4–6

Let A_1, G_1, and H_1 denote the arithmetic geometric and harmonic means, respectively, of two distinct positive numbers. For $n \ge 2$, let A_{n-1} and H_{n-1} have arithmetic, geometric and harmonic means as A_n, G_n, H_n, respectively. **(IIT-JEE 2007)**

4. Which one of the following statements is correct?

a. $G_1 > G_2 > G_3 > \cdots$

b. $G_1 < G_2 < G_3 < \cdots$

c. $G_1 = G_2 = G_3 = \cdots$

d. $G_1 < G_3 < G_5 < \cdots$ and $G_2 > G_4 > G_6 > \cdots$

5. Which one of the following statements is correct?

a. $A_1 > A_2 > A_3 > \cdots$

b. $A_1 < A_2 < A_3 < \cdots$

c. $A_1 > A_3 > A_5 > \cdots$ and $A_2 < A_4 < A_6 >$

d. $A_1 < A_3 < A_5 < \cdots$ and $A_2 > A_4 > A_6 > \cdots$

6. Which one of the following statements is correct?

a. $H_1 > H_2 > H_3 > \cdots$

b. $H_1 < H_2 < H_3 < \cdots$

c. $H_1 > H_3 > H_5 > \cdots$ and $H_2 < H_4 < H_6 < \cdots$

d. $H_1 < H_3 < H_6 < \cdots$ and $H_2 > H_4 > H_6 > \cdots$

Matching Column Type

1. Match the statements/expressions given in Column I with the values given in Column II.

Column – I	Column – II
(a) In R^2, if the magnitude of the projection vector of the vector $\alpha\hat{i} + \beta\hat{j}$ on $\sqrt{3}\hat{i} + \hat{j}$ is $\sqrt{3}$ and if $\alpha = 2 + \sqrt{3}\beta$, then possible value(s) of $\|\alpha\|$ is (are)	(p) 1
(b) Let a and b be real numbers such that the function $f(x) = \begin{cases} -3ax^2 - 2, & x < 1 \\ bx + a^2, & x \geq 1 \end{cases}$ is differentiable for all $x \in R$. Then possible value(s) of a is(are).	(q) 2
(c) Let $\omega \neq 1$ be a complex cube root of unity. If $(3 - 3\omega + 2\omega^2)^{4n+3} + (2 + 3\omega - 3\omega^2)^{4n+3} + (-3 + 2\omega + 3\omega^2)^{4n+3} = 0$, then possible value(s) of n is (are)	(r) 3
(d) Let the harmonic mean of two positive real numbers a and b be 4. If q is a positive real number such that $a, 5, q, b$ is an arithmetic progression, then the value(s) of $\|q - a\|$ is (are)	(s) 4
	(t) 5

(JEE Advanced 2015)

Integer Answer Type

1. Let S_k, $k = 1, 2, \ldots, 100$, denote the sum of the infinite geometric series whose first term is $\frac{k-1}{k!}$ and the common ratio is $\frac{1}{k}$, then the value of $\frac{100^2}{100!} + \sum_{k=1}^{100} |(k^2 - 3k + 1)S_k|$ is **(IIT-JEE 2010)**

2. Let $a_1, a_2, a_3, \ldots, a_{11}$ be real numbers satisfying $a_1 = 15, 27 - 2a_2 > 0$ and $a_k = 2a_{k-1} - a_{k-2}$ for $k = 3, 4, \ldots, 11$. If $\frac{a_1^2 + a_2^2 + \cdots + a_{11}^2}{11} = 90$, then the value of $\frac{a_1 + a_2 + \cdots + a_{11}}{11}$ is equal to **(IIT-JEE 2010)**

3. Let $a_1, a_2, a_3, \ldots, a_{100}$ be an arithmetic progression with $a_1 = 3$ and $S_p = \sum_{i=1}^{p} a_i, 1 \leq p \leq 100$. For any integer n with $1 \leq n \leq 20$, let $m = 5n$. If $\frac{S_m}{S_n}$ does not depend on n, then a_2 is **(IIT-JEE 2011)**

4. The minimum value of the sum of real numbers $a^{-5}, a^{-4}, 3a^{-3}, 1, a^8$, and a^{10} with $a > 0$ is **(IIT-JEE 2011)**

5. A pack contains n cards numbered from 1 to n. Two consecutive numbered cards are removed from the pack and the sum of the numbers on the remaining cards is 1224. If the smaller of the numbers on the removed cards is k, then $k - 20 =$ **(JEE Advanced 2013)**

6. Let a, b, c be positive integers such that $\frac{b}{a}$ is an integer. If a, b, c are in geometric progression and the arithmetic mean of a, b, c is $b + 2$, then the value of $\frac{a^2 + a - 14}{a + 1}$ is **(JEE Advanced 2014)**

7. Suppose that all the terms of an arithmetic progression (A.P.) are natural numbers. If the ratio of the sum of the first seven terms to the sum of the first eleven terms is 6 : 11 and the seventh term lies in between 130 and 140, then the common difference of this A.P. is **(JEE Advanced 2015)**

Assertion-Reasoning Type

1. Suppose four distinct positive numbers a_1, a_2, a_3, a_4 are in G.P. Let $b_1 = a_1, b_2 = b_1 + a_2, b_3 = b_2 + a_3$ and $b_4 = b_3 + a_4$.

   Statement 1: The numbers b_1, b_2, b_3, b_4 are neither in A.P. nor in G.P.

   Statement 2: The numbers b_1, b_2, b_3, b_4 are in H.P.

   a. Statement 1 is true, statement 2 is true; statement 2 is the correct explanation for statement 1.

   b. Statement 1 is true, statement 2 is true; statement 2 is not the correct explanation for statement 1.

   c. Statement 1 is true, statement 2 is false.

   d. Statement 1 is false, statement 2 is true.

   (IIT-JEE 2008)

Fill in the Blanks Type

1. The sum of integers from 1 to 100 that are divisible by 2 or 5 is ________. **(IIT-JEE 1984)**

2. The sum of the first n terms of the series $1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + 5^2 + 2 \times 6^2 + \cdots$ is $n(n+1)^2/2$, when n is even. When n is odd, the sum is ________. **(IIT-JEE 1988)**

3. Let the harmonic mean and geometric mean of two positive numbers be in the ratio 4 : 5. Then the two numbers are in the ratio ________. **(IIT-JEE 1992)**

4. For any odd integer $n \geq 1$, $n^3 - (n-1)^3 + \cdots + (-1)^{n-1} 1^3 =$ ________. **(IIT-JEE 1996)**

5. Let x be the arithmetic mean and y, z be the two geometric means between any two positive numbers. Then $\frac{y^3 + z^3}{xyz} =$ ________. **(IIT-JEE 1997)**

6. Let p and q be roots of the equation $x^2 - 2x + A = 0$ and let r and s be the roots of the equation $x^2 - 18x + B = 0$. If $p < q < r < s$ are in arithmetic progression, then $A =$ __________ and $B =$ __________. **(IIT-JEE 1997)**

True/False Type

1. For every integer $n > 1$, the inequality $(n!)^{1/n} < \frac{n+1}{2}$ holds. **(IIT-JEE 1981)**
2. If x and y are positive real numbers and m, n are any positive integers, then $\frac{x^n y^m}{(1+x^{2n})(1+y^{2m})} > \frac{1}{4}$. **(IIT-JEE 1989)**

Subjective Type

1. The interior angles of a polygon are in arithmetic progression. The smallest angle is 120°, and the common difference is 5°. Find the number of sides of the polygon. **(IIT-JEE 1980)**
2. Let x and y be two real numbers such that $x > 0$ and $xy = 1$. Find the minimum value of $x + y$. **(IIT-JEE 1981)**
3. If $a_1, a_2, \cdots, a_n$ are in arithmetic progression, where $a_i > 0$ for all i. Show that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \cdots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}} = \frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

(IIT-JEE 1982)

4. Does there exist a geometric progression containing 27, 8, and 12 as three of its terms? If it exists, how many such progressions are possible? **(IIT-JEE 1982)**
5. mn squares of equal size are arranged to from a rectangle of dimension m by n, where m and n are natural numbers. Two squares will be called 'neighbours' if they have exactly one common side. A natural number is written in each square such that the number written in any square is the arithmetic mean of the numbers written in its neighbouring squares. Show that this is possible only if all the numbers used are equal. **(IIT-JEE 1982)**
6. Find three numbers a, b, and c, between 2 and 18, such that
   (i) their sum is 25
   (ii) the numbers 2, a, b, are consecutive terms of an A.P. and
   (iii) the numbers b, c, and 18 are consecutive terms of a G.P. **(IIT-JEE 1983)**
7. The sum of the squares of three distinct real numbers, which are in G.P., is S^2. If their sum is aS, show that $a^2 \in \left(\frac{1}{3}, 1\right) \cup (1, 3)$. **(IIT-JEE 1986)**
8. If $\log_3 2$, $\log_3 (2^x - 5)$, and $\log_3 \left(2^x - \frac{7}{2}\right)$ are in arithmetic progression, determine the value of x. **(IIT-JEE 1990)**
9. Let p be the first of the n arithmetic means between two numbers and q the first of n harmonic means between the same numbers. Show that q does not lie between p and $\left(\frac{n+1}{n-1}\right)^2 p$. **(IIT-JEE 1991)**
10. If $S_1, S_2, S_3, \ldots, S_n$ are the sums of an infinite geometric series whose first terms are $1, 2, 3, \ldots, n$ and whose common ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n+1}$, respectively, then find the value of $S_1^2 + S_2^2 + S_3^2 + \cdots + S_{2n-1}^2$. **(IIT-JEE 1991)**
11. The harmonic mean of two numbers is 4. Their arithmetic mean A and the geometric mean G satisfy the relation $2A + G^2 = 27$. Find the two numbers. **(IIT-JEE 1997)**
12. Let a, b, c, and d be real numbers in a G.P. u, v, w, satisfy the system of equations
    $u + 2v + 3w = 6$
    $4u + 5v + 6w = 12$
    $6u + 9v = 4$
    Show that the roots of the equation $\left(\frac{1}{u}+\frac{1}{v}+\frac{1}{w}\right)x^2 + [(b-c)^2 + (c-a)^2 + (d-b)^2]\, x + u + v + w = 0$ and $20x^2 + 10(a-d)^2 x - 9 = 0$ are reciprocals of each other. **(IIT-JEE 1999)**
13. The fourth power of the common difference of an arithmetic progression with integer entries is added to the product of any four consecutive of it. Prove that the resulting sum is the squares of an integer. **(IIT-JEE 2000)**
14. Let a, b be positive real numbers. If a A_1, A_2, b be are in arithmetic progression, a, G_1, G_2, b are in geometric progression, and a, H_1, H_2, b are in harmonic progression, show that $\frac{G_1G_2}{H_1H_2} = \frac{A_1 + A_2}{H_1 + H_2} = \frac{(2a+b)(a+2b)}{9ab}$ **(IIT-JEE 2002)**
15. If a, b, c are in A.P. and a^2, b^2, c^2 are in H.P., then prove that either $a = b = c$ or a, b, $-\frac{c}{2}$ form a G.P. **(IIT-JEE 2003)**
16. If a, b, c are positive real numbers. Then prove that $(a+1)^7 (b+1)^7 (c+1)^7 > 7^7 a^4b^4c^4$. **(IIT-JEE 2003)**
17. Let $a_1, a_2, \ldots$ be positive real numbers in a geometric progression. For each n, let A_n, G_n, H_n be, respectively, the arithmetic mean, geometric mean, and harmonic mean of $a_1, a_2, \ldots, a_n$. Find an expression for the geomet-

ric mean of $G_1, G_2, \ldots, G_n$ in terms of $A_1, A_2, \ldots, A_n, H_1, H_2, \ldots, H_n$. **(IIT-JEE 2005)**

18. A cricketer plays n matches ($n \geq 1$), total number of runs scored by him is $\left(\frac{n+1}{4}\right)(2^{n+1} - n - 2)$. If he scores $k \cdot 2^{n-k+1}$ runs in kth match ($1 \leq k \leq n$), find the value of n. **(IIT-JEE 2005)**

19. If $a_n = \frac{3}{4} - \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \cdots + (-1)^{n-1}\left(\frac{3}{4}\right)^n$ and $b_n = 1 - a_n$, then find the least natural number n_0 such that $b_n > a_n$ $\forall\, n \geq n_0$. **(IIT-JEE 2006)**

Answer Key

JEE Advanced

Single Correct Answer Type

1. b. **2.** b. **3.** c. **4.** a. **5.** c.
6. c. **7.** c. **8.** d. **9.** d. **10.** c.
11. b. **12.** d. **13.** b. **14.** b. **15.** a.
16. d. **17.** d. **18.** a. **19.** c. **20.** d.
21. a. **22.** c. **23.** a. **24.** c. **25.** d.

Multiple Correct Answers Type

1. b., d. **2.** b., c. **3.** a., d. **4.** b., d. **5.** a., d.

Linked Comprehension Type

1. b. **2.** d. **3.** b. **4.** c. **5.** a.
6. b.

Matching Column Type

1. (d) – (q), (t)

Integer Answer Type

1. (3) **2.** (0) **3.** (6) **4.** (8) **5.** (5)
6. (4) **7.** (9)

Assertion–Reasoning Type

1. c.

Fill in the Blanks Type

1. 3050 **2.** $\frac{n^2(n+1)}{2}$ **3.** 1 : 4
4. $\frac{1}{4}(n+1)^2(2n-1)$ **5.** 2
6. $A = -3$ and $B = 77$

True/False Type

1. True **2.** False

Subjective Type

1. 9 **4.** infinite G.P. exists
6. $a = 5, b = 8, c = 12$ **8.** 3
10. $\frac{n(2n+1)(4n+1) - 3}{3}$ **11.** 3 and 6
17. $(A_1A_2 \ldots A_n \cdot H_1H_2 \ldots H_n)^{1/2n}$ **19.** 6

JEE Advanced

Single Correct Answer Type

1. b. $2\log_{10}x - \log_x 0.01 = 2\log_{10}x - \dfrac{\log_{10}0.01}{\log_{10}x}$

$$= 2\log_{10}x + \frac{2}{\log_{10}x}$$

$$= 2\left[\log_{10}x + \frac{1}{\log_{10}x}\right]$$

[Here $x > 1 \Rightarrow \log_{10}x > 0$]

Now,

A.M. $\geq$ G.M.

$$\Rightarrow \quad \frac{\log_{10}x + \dfrac{1}{\log_{10}x}}{2} \geq \left(\log_{10}x\,\frac{1}{\log_{10}x}\right)^{1/2}$$

or $\quad \log_{10}x + \dfrac{1}{\log_{10}x} \geq 2$

2. b. Given

$ar^2 = 4$

$\Rightarrow \quad a \times ar \times ar^2 \times ar^3 \times ar^4 = a^5 r^{10} = (ar^2)^5 = 4^5$

3. c. $2.\overline{357} = 2 + 0.357 + 0.000357 + \cdots \infty$

$$= 2 + \frac{357}{10^3} + \frac{357}{10^6} + \cdots \infty$$

$$= 2 + \frac{\dfrac{357}{10^3}}{1 - \dfrac{1}{10^3}}$$

$$= 2 + \frac{357}{999} = \frac{2355}{999}$$

Alternate solution:

Let

$x = 2.\overline{357}$

$\Rightarrow \quad 1000x = 2357.\overline{357}$

On subtracting, we get

$$999x = 2355 \text{ or } x = \frac{2355}{999}$$

4. a. For first equation $D = 4b^2 - 4ac = 0$ (as given a, b, c are in G.P.)

Thus, equation has equal roots which are equal to $-\dfrac{b}{a}$ each.

Thus, it should also be the root of the second equation. We have

$$d\left(\frac{-b}{a}\right)^2 + 2e\left(\frac{-b}{a}\right) + f = 0$$

or $\quad d\dfrac{b^2}{a^2} - 2\dfrac{be}{a} + f = 0$

or $\quad d\dfrac{ac}{a^2} - 2\dfrac{be}{a} + f = 0$ (as $b^2 = ac$)

or $\quad \dfrac{d}{a} + \dfrac{f}{c} = 2\dfrac{eb}{ac} = 2\dfrac{e}{b}$

5. c. Let

$$S=\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\cdots n \text{ terms}$$

$$=\left(1-\frac{1}{2}\right)+\left(1-\frac{1}{4}\right)+\left(1-\frac{1}{8}\right)+\left(1-\frac{1}{16}\right)+\cdots n\text{ terms}$$

$$=(1+1+1+\cdots n\text{ times})-\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots+\frac{1}{2^n}\right)$$

$$=n-\left[\frac{\frac{1}{2}\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}}\right]=n-1+2^{-n}$$

6. c. $S=(x+2)^{n-1}+(x+2)^{n-2}(x+1)+(x+2)^{n-3}(x+1)^2+\ldots+(x+1)^{n-1}$

Clearly the given series is G.P. of n terms with common ratio $\frac{x+1}{x+2}$

$$\therefore \quad S=\frac{(x+2)^{n-1}\left(1-\left(\frac{x+1}{x+2}\right)^n\right)}{1-\frac{x+1}{x+2}}$$

$$=(x+2)^n\left(1-\left(\frac{x+1}{x+2}\right)^n\right)$$

$$=(x+2)^n-(x+1)^n$$

7. c. Given that x, y, z are the pth, qth, and rth terms of an A.P.

$\therefore \quad x=A+(p-1)D$

$y=A+(q-1)D$

$z=A+(r-1)D$

$\Rightarrow \quad x-y=(p-q)D$

$y-z=(q-r)D$

$z-x=(r-p)D$

where A is the first term and D is the common difference. Also x, y, z are the pth, qth, and rth terms of a G.P.

$\therefore \quad x=aR^{p-1}, y=aR^{q-1}, z=aR^{r-1}$

$\therefore \quad x^{y-z}y^{z-x}z^{x-y}=(aR^{p-1})^{y-z}(aR^{q-1})^{z-x}(aR^{r-1})^{x-y}$

$=a^{y-z+z-x+x-y}R^{(p-1)(y-z)+(q-1)(z-x)+(r-1)(x-y)}$

$=A^{\circ}R^{(p-1)(q-r)D+(q-1)(r-p)D+(r-1)(p-q)D}$

$=A^{\circ}R^{\circ}=1$

8. d. Let $x_1, x_2, \ldots, x_n$ be the n positive numbers. Given that

$x_1x_2x_3\cdots x_n=1$ (1)

We know for positive numbers,

A.M. $\geq$ G.M.

$$\Rightarrow \quad \frac{x_1+x_2+\cdots+x_n}{n}\geq\sqrt[n]{x_1x_2\cdots x_n}$$

or $\frac{x_1+x_2+\cdots+x_n}{n}\geq 1$ [Using Eq. (1)]

or $x_1+x_2+\cdots+x_n\geq n$

9. d. $\ln(a+c)$, $\ln(a-c)$, $\ln(a-2b+c)$ are in A.P. Hence, $a+c$, $c-a$, $a-2b+c$ are in G.P. Therefore,

$(a-c)^2=(a+c)(a-2b+c)$

or $(a-c)^2=(a+c)^2-2b(a+c)$

or $2b(a+c)=(a+c)^2-(a-c)^2$

or $2b(a+c)=4ac$

or $b=\frac{2ac}{a+c}$

Hence, a, b, and c are in H.P.

10. c. $T_m=a+(m-1)d=1/n$

$T_n=a+(n-1)d=1/m$

$\therefore \quad (m-n)d=1/n-1/m=(m-n)/mn$

or $d=1/mn$

$\Rightarrow \quad a=\frac{1}{mn}$

$\therefore \quad T_{mn}=a+(mn-1)d$

$=\frac{1}{mn}+(mn-1)\frac{1}{mn}$

$=\frac{1}{mn}+1-\frac{1}{mn}$

$=1$

11. b. If x, y, and z are in G.P. ($x, y, z>1$), then $\log x$, $\log y$, $\log z$ are in A.P. Hence,

$1+\log x$, $1+\log y$, $1+\log z$ will also be in A.P.

$\Rightarrow \quad \frac{1}{1+\log x}, \frac{1}{1+\log y}, \frac{1}{1+\log z}$ will be in H.P.

12. d. $a_1=h_1=2, a_{10}=h_{10}=3$

$3=a_{10}=2+9d\Rightarrow d=1/9$

$\therefore \quad a_4=2+3d=7/3$

Also,

$3=h_{10}$ or $\frac{1}{3}=\frac{1}{h_{10}}=\frac{1}{2}+9D$

or $D=-\frac{1}{54}$

$\Rightarrow \quad \frac{1}{h_7}=\frac{1}{2}+6D=\frac{1}{2}-\frac{1}{9}=\frac{7}{18}$

$\therefore \quad a_4h_7=\frac{7}{3}\times\frac{18}{7}=6$

13. b. Harmonic mean H of roots α and β is

$$H=\frac{2\alpha\beta}{\alpha+\beta}=\frac{2\frac{8+2\sqrt{5}}{5+\sqrt{2}}}{\frac{4+\sqrt{5}}{5+\sqrt{2}}}=4$$

14. b. Since A.M. > G.M. for different numbers, so

$$\frac{(b+c-a)+(c+a-b)}{2}>[(b+c-a)(c+a-b)]^{1/2}$$

$\Rightarrow \quad c>[(b+c-a)(c+a-b)]^{1/2}$

Similarly,

$b>[(b+c-a)(a+b-c)]^{1/2}$

and

$a>[(a+b-c)(c+a-b)]^{1/2}$

Multiplying, we get

$abc>(b+c-a)(c+a-b)(a+b-c)$

$\Rightarrow \quad (b+c-a)(c+a-b)(a+b-c)-abc<0$

15. a. As A.M. ≥ G.M. for positive real numbers, we get

$$\frac{(a+b)+(c+d)}{2} \geq \sqrt{(a+b)(c+d)}$$

$\Rightarrow \quad M \leq 1$

Also,

$(a+b)(c+d) > 0 \quad [\because a, b, c, d > 0]$

$\therefore \quad 0 \leq M \leq 1$

16. d. a, b, and c, d are in A.P. Therefore, d, c, b and a are also in A.P. Hence,

$\frac{d}{abcd}, \frac{c}{abcd}, \frac{b}{abcd}, \frac{a}{abcd}$ are also in A.P.

$\Rightarrow \quad \frac{1}{abc}, \frac{1}{abd}, \frac{1}{acd}, \frac{1}{bcd}$ are in A.P.

$\Rightarrow \quad abc, abd, acd, bcd$ are in H.P.

17. d. Sum is 4 and second term is 3/4. It is given that the first term is a and common ratio is r. Hence,

$\frac{a}{1-r} = 4$ and $ar = \frac{3}{4}$ or $r = \frac{3}{4a}$

Therefore,

$\frac{a}{1-\frac{3}{4a}} = 4$ or $\frac{4a^2}{4a-3} = 4$

or $\quad a^2 - 4a + 3 = 0$

or $\quad (a-1)(a-3) = 0$

$\Rightarrow \quad a = 1$ or 3

When $a = 1$, $r = 3/4$ and when $a = 3$, $r = 1/4$.

18. a. α, β are the roots of $x^2 - x + p = 0$. Hence,

$\alpha + \beta = 1$ (1)

$\alpha\beta = p$ (2)

γ, δ are the roots of $x^2 - 4x + q = 0$. Hence,

$\gamma + \delta = 4$ (3)

$\gamma\delta = q$ (4)

$\alpha, \beta, \gamma, \delta$ are in G.P. Let $\alpha = a$, $\beta = ar$, $\gamma = ar^2$, $\delta = ar^3$. Substituting these values in Eqs. (1), (2), (3), and (4), we get

$a + ar = 1$ (5)

$a^2r = p$ (6)

$ar^2 + ar^3 = 4$ (7)

$a^2r^5 = q$ (8)

Dividing (7) by (5), we get

$\frac{ar^2(1+r)}{a(1+r)} = \frac{4}{1}$ or $r^2 = 4$ or $r = 2, -2$

from (5) $a = \frac{1}{1+r} = \frac{1}{1+2}$ or $\frac{1}{1-2} = \frac{1}{3}$ or -1

As p is an integer (given), r is also an integer (2 or –2). Therefore, from (6), $a \neq 1/3$. Hence, $a = -1$ and $r = -2$.

$\therefore \quad p = (-1)^2 \times (-2) = -2$

$q = (-1)^2 \times (-2)^5 = -32$

19. c. Given,

$2 + 5 + 8 + \cdots 2n$ terms $= 57 + 59 + 61 + \cdots n$ terms

$\Rightarrow \quad \frac{2n}{2}[4 + (2n-1)3] = \frac{n}{2}[114 + (n-1)2]$

or $\quad 6n + 1 = n + 56$

or $\quad 5n = 55$

or $\quad n = 11$

20. d. Given that a, b, and c are in A.P. Hence,

$2b = a + c$

But given,

$a + b + c = 3/2$

or $\quad 3b = 3/2$

or $\quad b = 1/2$

Hence,

$a + c = 1$

Again, a^2, b^2, c^2 are in G.P. Hence,

$b^4 = a^2c^2$

or $\quad b^2 = \pm ac$

$\Rightarrow \quad ac = \frac{1}{4}$ or $-\frac{1}{4}$ and $a + c = 1$ (1)

Now,

$a + c = 1$ and $ac = \frac{1}{4}$

$\Rightarrow \quad (a-c)^2 = (a+c)^2 - 4ac = 1 - 1 = 0$

$\Rightarrow \quad a = c$

But $a \neq c$ as given that $a < b < c$. We consider $a + c = 1$ and $ac = -1/4$. Hence,

$(a-c)^2 = 1 + 1 = 2$

or $\quad a - c = \pm\sqrt{2}$

But

$a < c \therefore a - c = -\sqrt{2}$ (2)

Solving (1) and (2), we get

$a = \frac{1}{2} - \frac{1}{\sqrt{2}}$

21. a. From A.M. ≥ G.M., we have

$$\frac{(a_1 + a_2 + \cdots + a_{n-1} + 2a_n)}{n} \geq (a_1a_2\cdots a_{n-1}2a_n)^{1/n}$$

or $\quad \frac{(a_1 + a_2 + \cdots + a_{n-1} + 2a_n)}{n} \geq (2c)^{1/n}$

or $\quad a_1 + a_2 + \cdots + a_{n-1} + 2a_n \geq n(2c)^{1/n}$

22. c. $S_\infty = \frac{a}{1-r} = 5$ (given)

$\Rightarrow \quad r = \frac{5-a}{5}$

But

$0 < |r| < 1$

$\Rightarrow \quad 0 < \left|\frac{5-a}{5}\right| < 1$

or $\quad -1 < \frac{5-a}{5} < 1$ and $a \neq 5$

or $\quad -5 < 5 - a < 5$ and $a \neq 5$

or $\quad -10 < -a < 0$ and $a \neq 5$

or $\quad 10 > a > 0$ and $a \neq 5$

or $\quad 0 < a < 10$ and $a \neq 5$

23. a. Using A.M. ≥ G.M., we have

$$\frac{\sqrt{x^2+x} + \frac{\tan^2\alpha}{\sqrt{x^2+x}}}{2} \geq \left(\sqrt{x^2+x}\,\frac{\tan^2\alpha}{\sqrt{x^2+x}}\right)^{1/2}$$

or $\quad \sqrt{x^2+x} + \frac{\tan^2\alpha}{\sqrt{x^2+x}} \geq 2\tan\alpha \qquad \left(\because \alpha \in \left(0, \frac{\pi}{2}\right)\right)$

24. c. We have $S_n = cn^2$

$$\therefore \quad t_n = S_n - S_{n-1} = c\{n^2 - (n-1)^2\} = c(2n-1)$$

$$\Rightarrow \quad t_n^2 = c^2(4n^2 - 4n + 1)$$

$$\Rightarrow \quad \sum_{n=1}^{n} t_n^2 = c^2\left\{\frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n\right\}$$

$$= \frac{c^2 n}{3}\{(2n+2)(2n+1) - 6(n+1) + 3\}$$

$$= \frac{c^2}{3} n(4n^2 - 1)$$

25. d. $a_1, a_2, a_3, \ldots$ are in H.P.

$$\Rightarrow \quad \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots \text{ are in A.P.}$$

$$\Rightarrow \quad \frac{1}{a_n} = \frac{1}{a_1} + (n-1)d < 0, \text{ where } \frac{\frac{1}{25} - \frac{5}{25}}{19} = d = \left(\frac{-4}{19 \times 25}\right)$$

$$\Rightarrow \quad \frac{1}{5} + (n-1)\left(\frac{-4}{19 \times 25}\right) < 0$$

or $\quad \dfrac{4(n-1)}{19 \times 5} > 1$

or $\quad n - 1 > \dfrac{19 \times 5}{4}$

or $\quad n > \dfrac{19 \times 5}{4} + 1$

or $\quad n \geq 25.$

Multiple Correct Answers Type

1. b., d.

Let x be the first term and y be the $(2n-1)$th term of A.P., G.P. and H.P. whose nth terms are a, b, c, respectively. Now according to the property of A.P., G.P., and H.P., x, a, y are in A.P.; x, b, y are in G.P. and x, c, y are in H.P. Hence,

$$a = \frac{x+y}{2} = \text{A.M.}$$

$$b = \sqrt{xy} = \text{G.M.}$$

$$c = \frac{2xy}{x+y} = \text{H.M.}$$

Now, A.M, G.M. and H.M. are in G.P. Hence,

$$b^2 = ac$$

Also, A.M. ≥ G.M. ≥ H.M. Hence,

$$a \geq b \geq c$$

2. b., c.

We have, for $0 < \phi < \pi/2$

$$x = \sum_{n=0}^{\infty} \cos^{2n}\phi = 1 + \cos^2\phi + \cos^4\phi + \cdots \infty = \frac{1}{1-\cos^2\phi}$$

$$= \frac{1}{\sin^2\phi} \qquad (1)$$

$$y = \sum_{n=0}^{\infty} \sin^{2n}\phi = 1 + \sin^2\phi + \sin^4\phi + \cdots \infty = \frac{1}{1-\sin^2\phi}$$

$$= \frac{1}{\cos^2\phi} \qquad (2)$$

$$z = \sum_{n=0}^{\infty} \cos^{2n}\phi \sin^{2n}\phi = 1 + \cos^2\phi\sin^2\phi + \cos^4\phi\sin^4\phi + \cdots \infty$$

$$= \frac{1}{1-\cos^2\phi\sin^2\phi} \qquad (3)$$

Substituting the values of $\cos^2\phi$ and $\sin^2\phi$ in (3) from (1) and (2), we get

$$z = \frac{1}{1 - \frac{1}{x}\frac{1}{y}}$$

or $\quad z = \dfrac{xy}{xy-1}$

or $\quad xyz - z = xy$

or $\quad xyz = xy + z$

Also,

$$x + y + z = \frac{1}{\cos^2\phi} + \frac{1}{\sin^2\phi} + \frac{1}{1-\cos^2\phi\sin^2\phi}$$

$$= \frac{(\sin^2\phi + \cos^2\phi)(1-\cos^2\phi\sin^2\phi) + \cos^2\phi\sin^2\phi}{\cos^2\phi\sin^2\phi(1-\cos^2\phi\sin^2\phi)}$$

$$= \frac{(1-\cos^2\phi\sin^2\phi) + \cos^2\phi\sin^2\phi}{\cos^2\phi\sin^2\phi(1-\cos^2\phi\sin^2\phi)}$$

$$= \frac{1}{\cos^2\phi\sin^2\phi(1-\cos^2\phi\sin^2\phi)} = xyz$$

Thus, (b) and (c) both are correct.

3. a., d.

We have

$$a(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{2^n - 1}$$

$$= 1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{15}\right) + \cdots + \frac{1}{2^n - 1}$$

$$= 1 + \left(\frac{1}{2} + \frac{1}{2^2-1}\right) + \left(\frac{1}{2^2} + \frac{1}{5} + \frac{1}{6} + \frac{1}{2^3-1}\right) + \left(\frac{1}{2^3} + \cdots + \frac{1}{2^4-1}\right) + \cdots$$

$$< 1 + 1 + \cdots + 1 = n$$

Thus,

$$a(100) < 100$$

Also,

$$a(n) = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \frac{1}{2^n - 1}$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{2^1+1} + \frac{1}{2^2}\right) + \left(\frac{1}{2^2+1} + \frac{1}{2^3}\right) + \cdots$$

$$+ \left(\frac{1}{2^{n-1}+1}\right)$$

$$> 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \cdots + \frac{2^{n-1}}{2^n} - \frac{1}{2^n}$$

$$= 1 + \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}\right) - \frac{1}{2^n}$$

$$= 1 + \frac{n}{2} - \frac{1}{2^n} = \left(1 - \frac{1}{2^n}\right) + \frac{n}{2}$$

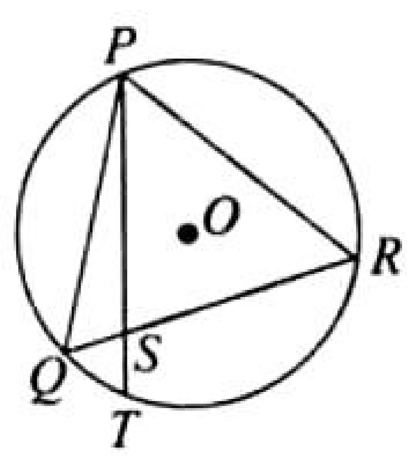
Thus,

$$a(200) > \left(1 - \frac{1}{2^{200}}\right) + \frac{200}{2} > 100$$

i.e.,

$$a(200) > 100$$

4. b., d.



We have PS × ST = QS × SR (property of circle). Now,

A.M. > G.M.

$$\Rightarrow \quad \frac{\frac{1}{PS} + \frac{1}{ST}}{2} > \sqrt{\frac{1}{PS} \times \frac{1}{ST}}$$

$$\text{or} \quad \frac{1}{PS} + \frac{1}{ST} > \frac{2}{\sqrt{QS \times SR}}$$

Also,

$$\frac{QS + SR}{2} > \sqrt{QS \times SR}$$

$$\Rightarrow \quad \frac{1}{\sqrt{QS \times SR}} > \frac{2}{QR}$$

$$\text{or} \quad \frac{1}{PS} + \frac{1}{ST} > \frac{4}{QR}$$

5. a., d.

$$S_n = \sum_{k=1}^{4n} (-1)^{\frac{k(k+1)}{2}} k^2$$

$$= -1^2 - 2^2 + 3^2 + 4^2 - 5^2 - 6^2 + 7^2 + 8^2 \ldots\ldots\ldots$$
$$- (4n-3)^2 - (4n-2)^2 + (4n-1)^2 + (4n)^2$$

$$= \sum_{r=1}^{n} \left[(4r)^2 + (4r-1)^2 - (4r-2)^2 - (4r-3)^2\right]$$

$$= \sum_{r=1}^{n} \left[((4r)^2 - (4r-2)^2) + ((4r-1)^2 - (4r-3)^2)\right]$$

$$= \sum_{r=1}^{n} [2(8r-2) + 2(8r-4)]$$

$$= \sum_{r=1}^{n} [32r - 12]$$

$$= 16n(n+1) - 12n$$

$$= 4n(4n+1)$$

$$= \begin{cases} 1056 \text{ for } n = 8 \\ 1332 \text{ for } n = 9 \end{cases}$$

Linked Comprehension Type

1. b. $V_1 + V_2 + \cdots + V_n$

$$= \sum_{r=1}^{n} V_r$$

$$= \sum_{r=1}^{n} \left(\frac{r}{2}(2r + (r-1)(2r-1))\right)$$

$$= \sum_{r=1}^{n} \left(r^3 - \frac{r^2}{2} + \frac{r}{2}\right)$$

$$= \Sigma n^3 - \frac{\Sigma n^2}{2} + \frac{\Sigma n}{2}$$

$$= \frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{4}$$

$$= \frac{n(n+1)}{4}\left[n(n+1) - \frac{2n+1}{3} + 1\right]$$

$$= \frac{n(n+1)(3n^2+n+2)}{12}$$

2. d. $T_r = V_{r+1} - V_r - 2$

$$= \left[(r+1)^3 - \frac{(r+1)^2}{2} + \frac{(r+1)}{2}\right] - \left[r^3 - \frac{r^2}{2} + \frac{r}{2}\right] - 2$$

$$= (r+1)(3r-1)$$

For each r, T_r has two different factors other than 1 and itself. Therefore, T_r is always a composite number.

3. b. $Q_r = T_{r+1} - T_r$

$$= (r+2)(3r+2) - (r+1)(3r-1)$$

$$= 6r + 5$$

Since $Q_{r+1} - Q_r = 6(r+1) + 5 - 6r - 5 = 6$ (constant), therefore, $Q_1, Q_2, Q_3, \ldots$ are in A.P. with common difference 6.

4. c. Given,

$$A_1 = \frac{a+b}{2},\ G_1 = \sqrt{ab},\ H_1 = \frac{2ab}{a+b}$$

Also,

$$A_n = \frac{A_{n-1} + H_{n-1}}{2}$$

$$G_n = \sqrt{A_{n-1} H_{n-1}}$$

$$H_n = \frac{2A_{n-1} \times H_{n-1}}{A_{n-1} + H_{n-1}}$$

$$\Rightarrow \quad G_n^2 = A_n H_n \Rightarrow A_n H_n = A_{n-1} H_{n-1}$$

Similarly, we can prove

$$A_n H_n = A_{n-1} H_{n-1} = A_{n-2} H_{n-2} = \cdots = A_1 H_1$$

$$\Rightarrow \quad A_n H_n = ab$$

$$\Rightarrow \quad G_1^2 = G_2^2 = G_3^2 = \cdots = G_n^2 = ab$$

or $\quad G_1 = G_2 = G_3 = \cdots = \sqrt{ab}$

5. a. We have

$$A_n = \frac{A_{n-1} + H_{n-1}}{2}$$

$$\therefore \quad A_n - A_{n-1} = \frac{A_{n-1} + H_{n-1}}{2} - A_{n-1}$$

$$= \frac{H_{n-1} - A_{n-1}}{2} < 0 \qquad (\because A_{n-1} > H_{n-1})$$

$$\Rightarrow \quad A_n < A_{n-1} \text{ or } A_{n-1} > A_n$$

Hence, we can conclude that $A_1 > A_2 > A_3 > \cdots$

6. b. We have

$$A_n H_n = ab \text{ or } H_n = \frac{ab}{A_n}$$

$$\frac{1}{A_{n-1}} < \frac{1}{A_n} \Rightarrow H_{n-1} < H_n$$

$$\therefore \quad H_1 < H_2 < H_3 < \cdots$$

Matching Column Type

1. (d) – (q), (t)

$$\frac{2ab}{a+b} = 4$$

$$\Rightarrow \quad ab = 2a + 2b \qquad (1)$$

Now, $q = 10 - a$

and $\quad 2q = 5 + b$

$\Rightarrow \quad 20 - 2a = 5 + b$

$\Rightarrow \quad 15 = 2a + b \qquad (2)$

From (1) and (2) $a(15 - 2a) = 2a + 2(15 - 2a)$

$\Rightarrow \quad 15a - 2a^2 = -2a + 30$

$\Rightarrow \quad 2a^2 - 17a + 30 = 0$

$\Rightarrow \quad a = 6, \frac{5}{2}$ accordingly $q = 4, \frac{15}{2}$

$\Rightarrow \quad |q - a| = 2, 5$

Note: Solutions of the remaining parts are given in their respective chapters.

Integer Answer Type

1. (3) $S_k = \dfrac{\frac{k-1}{k!}}{1 - \frac{1}{k}} = \dfrac{1}{(k-1)!}$

$$\sum_{k=2}^{100} \left| (k^2 - 3k + 1)\frac{1}{(k-1)!} \right|$$

$$= \sum_{k=2}^{100} \left| \frac{(k-1)^2 - k}{(k-1)!} \right|$$

$$= \sum_{k=2}^{100} \left| \frac{k-1}{(k-2)!} - \frac{k}{(k-1)!} \right|$$

$$= \left| \frac{1}{0!} - \frac{2}{1!} \right| + \left| \frac{2}{1!} - \frac{3}{2!} \right| + \left| \frac{3}{2!} - \frac{4}{3!} \right| + \cdots$$

$$= \frac{2}{1!} - \frac{1}{0!} + \frac{2}{1!} - \frac{3}{2!} + \frac{3}{2!} - \frac{4}{3!} + \cdots + \frac{99}{98!} - \frac{100}{99!}$$

$$= 3 - \frac{100}{99!}$$

2. (0) $a_k = 2a_{k-1} - a_{k-2} \Rightarrow a_1, a_2, \ldots, a_{11}$ are in A.P.

$$\therefore \quad \frac{a_1^2 + a_2^2 + \cdots + a_{11}^2}{11} = \frac{11a^2 + 35 \times 11d^2 + 110ad}{11} = 90$$

$$\Rightarrow \quad 225 + 35d^2 + 150d = 90$$

$$35d^2 + 150d + 135 = 0 \quad \Rightarrow \quad d = -3, -9/7$$

Given $a_2 < \frac{27}{2}$, we get $d = -3$ and $d = -9/7$

$$\Rightarrow \quad \frac{a_1 + a_2 + \cdots + a_{11}}{11} = \frac{11}{2}[30 - 10 \times 3] = 0$$

3. (6) $a_1, a_2, a_3, \ldots, a_{100}$ is an A.P.

$$a_1 = 3, S_p = \sum_{i=1}^{p} a_i, 1 \le p \le 100$$

$$\frac{S_m}{S_n} = \frac{S_{5n}}{S_n} = \frac{\frac{5n}{2}(6 + (5n-1)d)}{\frac{n}{2}(6 + (n-1)d)}$$

$$= \frac{5((6-d) + 5nd)}{6 - d + nd}$$

$\frac{S_m}{S_n}$ is independent of n if $6 - d = 0 \quad \Rightarrow \quad d = 6.$

4. (8) Using A.M. ≥ G.M.

$$\Rightarrow \quad \frac{a^{-5} + a^{-4} + a^{-3} + a^{-3} + a^{-3} + a^8 + a^{10} + 1}{8} \ge 1$$

or $\quad a^{-5} + a^{-4} + a^{-3} + a^{-3} + a^{-3} + a^8 + a^{10} + 1 \ge 8$

or $\quad \left(a^{-5} + a^{-4} + 3a^{-3} + a^8 + a^{10} + 1\right)_{\min} = 8$

5. (5) Clearly, $1 + 2 + 3 + \ldots + n - 2 \le 1224 \le 3 + 4 + \ldots n$

$$\Rightarrow \quad \frac{(n-2)(n-1)}{2} \le 1224 \le \frac{(n-2)}{2}(3+n)$$

$\Rightarrow \quad n^2 - 3n - 2446 \le 0$ and $n^2 + n - 2454 \ge 0$

$\Rightarrow \quad 49 < n < 51 \Rightarrow n = 50$

$$\therefore \quad \frac{n(n+1)}{2} - (2k+1) = 1224$$

or $\quad k = 25$

or $\quad k - 20 = 5$

6. (4) According to the question

$$\frac{b}{a} = \frac{c}{b} = \text{(integer)}$$

$$\Rightarrow \quad b^2 = ac \Rightarrow c = \frac{b^2}{a}$$

Also given $\frac{a+b+c}{3} = b + 2$

$\Rightarrow \quad a + b + c = 3b + 6$

$\Rightarrow \quad a - 2b + c = 6$

$$\Rightarrow \quad a - 2b + \frac{b^2}{a} = 6$$

$$\Rightarrow \quad 1 - \frac{2b}{a} + \frac{b^2}{a^2} = \frac{6}{a}$$

$$\Rightarrow \quad \left(\frac{b}{a} - 1\right)^2 = \frac{6}{a} \qquad \left(\because \frac{b}{a} \text{ is an integer}\right)$$

$\Rightarrow \quad a = 6$ only

$$\Rightarrow \quad \frac{a^2+a-14}{a+1}=4$$

7. (9) Given $\frac{S_7}{S_{11}}=\frac{6}{11}$

$$\Rightarrow \quad \frac{\frac{7}{2}[2a+6d]}{\frac{11}{2}[2a+10d]}=\frac{6}{11} \qquad \text{(Given } 130 < a+6d < 140)$$

$$\Rightarrow \quad \frac{7(a+3d)}{11(a+5d)}=\frac{6}{11}$$

$\Rightarrow \quad 7a+21d=6a+30d$

$\Rightarrow \quad a=9d$

$\Rightarrow \quad T_7=a+6d=15d$

Given $130 < T_7 < 140$

$\Rightarrow \quad 130 < 15d < 140$

$\Rightarrow \quad d=9$

Assertion-Reasoning Type

1. c. $b_1=a_1, b_2=a_1+a_2 \cdot b_3=a_1+a_2+a_3 \cdot b_4=a_1+a_2+a_3+a_4$
Hence b_1, b_2, b_3, b_4 are neither in A.P. nor in G.P. nor in H.P.

Fill in the Blanks Type

1. The sum of integers from 1 to 100 that are divisible by 2 or 5 is

S = sum of integers from 1 to 100 divisible by 2
+ sum of integers from 1 to 100 divisible by 5
− sum of integers from 1 to 100 divisible by 10

$$=(2+4+6+\cdots+100)+(5+10+15+\cdots+100)-(10+20+\cdots+100)$$

$$=\frac{50}{2}[2\times 2+49\times 2]+\frac{20}{2}[2\times 5+19\times 5]-\frac{10}{2}[2\times 10+9\times 10]$$

$$=2550+1050-550=3050$$

2. When n is odd, last term is n^2. Hence, the required sum is

$$S=[1^2+2\times 2^2+3^2+2\times 4^2+\cdots+2\times(n-1)^2]+n^2$$

$$=\frac{(n-1)\,n^2}{2}+n^2 \qquad \text{[Using sum for } (n-1) \text{ to be even]}$$

$$=\frac{n^2(n+1)}{2}$$

3. Let a and b be two positive numbers. Then, H.M. $=\frac{2ab}{a+b}$ and G.M. $=\sqrt{ab}$. According to question, H.M.:G.M. = 4:5

$$\therefore \quad \frac{2ab}{(a+b)\sqrt{ab}}=\frac{4}{5}$$

or $$\frac{2\sqrt{ab}}{a+b}=\frac{4}{5}$$

or $$\frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}}=\frac{5+4}{5-4}$$

or $$\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\right)^2=9$$

or $$\frac{(\sqrt{a}+\sqrt{b})}{\sqrt{a}-\sqrt{b}}=3,-3$$

or $$\frac{2\sqrt{a}}{2\sqrt{b}}=\frac{3+1}{3-1},\frac{-3+1}{-3-1}$$

or $$\frac{\sqrt{a}}{\sqrt{b}}=2,\frac{1}{2} \text{ or } \frac{a}{b}=4,\frac{1}{4}$$

$\Rightarrow \quad a:b=4:1$ or $1:4$

4. Since n is an odd integer, $(-1)^{n-1}=1$ and $n-1, n-3, n-5, \ldots$ are even integers. The given series is

$$n^3-(n-1)^3+(n-2)^3-(n-3)^3+\cdots+(-1)^{n-1}1^3$$

$$=1^3-2^3+3^3-4^3+\ldots\ldots-(n-1)^3+n^3$$

(writing series in reverse order)

$$=(1^3+2^3+3^3+4^3+\ldots\ldots+(n-1)^3+n^3)-2(2^3+4^3+\ldots..(n-1)^3)$$

$$=(1^3+2^3+3^3+4^3+\ldots\ldots+(n-1)^3+n^3)-16(1^3+2^3+\ldots..((n-1)/2)^3)$$

$$=\left[\frac{n(n+1)}{2}\right]^2-16\left[\left\{\frac{1}{2}\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}+1\right)\right\}\right]^2$$

$$=\frac{1}{4}n^2(n+1)^2-16\,\frac{(n-1)^2(n+1)^2}{16\times 4}$$

$$=\frac{1}{4}(n+1)^2[n^2-(n-1)^2]$$

$$=\frac{1}{4}(n+1)^2(2n-1)$$

5. Given that x is the A.M. between a and b and y, and z are the G.M.'s between a and b where a and b are positive. Then a, x, b are in A.P. So,

$$x=\frac{a+b}{2}$$

a, y, z, b are in G.P. So,

$y=ar$, and $z=ar^2$, where $r=\sqrt[3]{\frac{b}{a}}$. Also, $yz=ab$. Now,

$$\frac{y^3+z^3}{xyz}=\frac{a^3r^3+a^3r^6}{ab\left(\frac{a+b}{2}\right)}$$

$$=\frac{a^3\times\frac{b}{a}+a^3\times\frac{b^2}{a^2}}{ab\left(\frac{a+b}{2}\right)}$$

$$=\frac{2(a^2b+ab^2)}{a^2b+ab^2}$$

$$=2$$

6. Let p and q be roots of the equation $x^2-2x+A=0$. Then,

$p+q=2, pq=A$

Let r and s be the roots of the equation $x^2-18x+B=0$. Then,

$r+s=18, rs=B$

And it is given that p, q, r, and s are in A.P. Let $p = a - 3d$, $q = a - d$, $r = a + d$, and $s = a + 3d$. As $p < q < r < s$, we have $d > 0$. Now,

$$2 = p + q = a - 3d + a - d = 2a - 4d$$

or $a - 2d = 1$ (1)

and

$$18 = r + s = a + d + a + 3d$$

or $a + 2d = 9$ (2)

Solving (1) and (2), $a = 5, d = 2$

$\therefore$ $p = -1, q = 3, r = 7, s = 11$

Therefore, $A = pq = -3$ and $B = rs = 77$.

True/False Type

1. **True** Consider n numbers, 1, 2, 3, 4, ..., n. Now, using A.M. > G.M. for distinct numbers, we get

$$\frac{1+2+3+4+\cdots+n}{n} > (1\times 2\times 3\times 4\times\cdots\times n)^{1/n}$$

$$\Rightarrow \frac{n(n+1)}{2n} > (n!)^{1/n}$$

or $(n!)^{1/n} < \frac{n+1}{2}$

Hence, the statement is true.

2. **False** As x and y are positive real numbers and m and n are positive integers, we have

$$\frac{1+x^{2n}}{2} \ge (1\times x^{2n})^{1/2}$$

and

$$\frac{1+y^{2m}}{2} \ge (1\times y^{2m})^{1/2}$$

(since for two positive numbers A.M. ≥ G.M.)

$$\therefore \left(\frac{1+x^{2n}}{2}\right) \ge x^n \quad (1)$$

and

$$\left(\frac{1+y^{2m}}{2}\right) \ge y^m \quad (2)$$

Multiplying Eqs. (1) and (2), we get

$$\frac{(1+x^{2n})(1+y^{2m})}{4} \ge x^n y^m$$

or $\frac{1}{4} \ge \frac{x^n y^m}{(1+x^{2n})(1+y^{2m})}$

Hence, the statement is false.

Subjective Type

1. Let there be n sides in the polygon. Then by geometry, the sum of all n interior angles of polygon is $(n-2)\times 180°$. Also the angles are in A.P. with the smallest angle 120° and common difference 5°.

Therefore, sum of all interior angles of polygon is

$$\frac{n}{2}[2\times 120 + (n-1)\times 5]$$

Thus, we must have

$$\frac{n}{2}[2\times 120 + (n-1)5] = (n-2)\times 180$$

or $\frac{n}{2}[5n + 235] = (n-2)\times 180$

or $5n^2 + 235n = 360n - 720$

or $5n^2 - 125n + 720 = 0$

or $n^2 - 25n + 144 = 0$

or $(n-16)(n-9) = 0$

or $n = 16, 9$

But if $n = 16$, then 16th angle $= 120 + 15\times 5 = 195 > 180°$ which is not possible. Hence, $n = 9$.

2. $x > 0$ and $xy = 1$

$\therefore$ $y > 0$

Now, A.M. ≥ G.M.

$$\Rightarrow \frac{x+y}{2} \ge (xy)^{\frac{1}{2}}$$

$$\Rightarrow \frac{x+y}{2} \ge 1$$

$$\Rightarrow x + y \ge 2$$

Thus, least value of $x + y$ is 2.

3. Given, $a_1, a_2, \ldots, a_n$ are in A.P., $\forall\, a_i > 0$.

$\therefore$ $a_1 - a_2 = a_2 - a_3 = \cdots = a_{n-1} - a_n = d$ (a constant)

Now,

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \cdots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}$$

$$= \frac{\sqrt{a_1}-\sqrt{a_2}}{a_1-a_2} + \frac{\sqrt{a_2}-\sqrt{a_3}}{a_2-a_3} + \cdots + \frac{\sqrt{a_{n-1}}-\sqrt{a_n}}{a_{n-1}-a_n}$$

$$= \frac{1}{-d}\left[\sqrt{a_1}-\sqrt{a_n}\right]$$

$$= \frac{a_1 - a_n}{-d(\sqrt{a_1}+\sqrt{a_n})}$$

$$= \frac{(n-1)d}{d(\sqrt{a_1}+\sqrt{a_n})}$$

$$= \frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

4. If possible let 8 be the first term and 12 and 27 be mth and nth terms of G.P., respectively.

$\Rightarrow$ $12 = ar^{m-1} = 8r^{m-1}$, and $27 = 8r^{n-1}$

$\Rightarrow$ $\frac{3}{2} = r^{m-1}$, and $\left(\frac{3}{2}\right)^3 = r^{n-1} = r^{3(m-1)}$

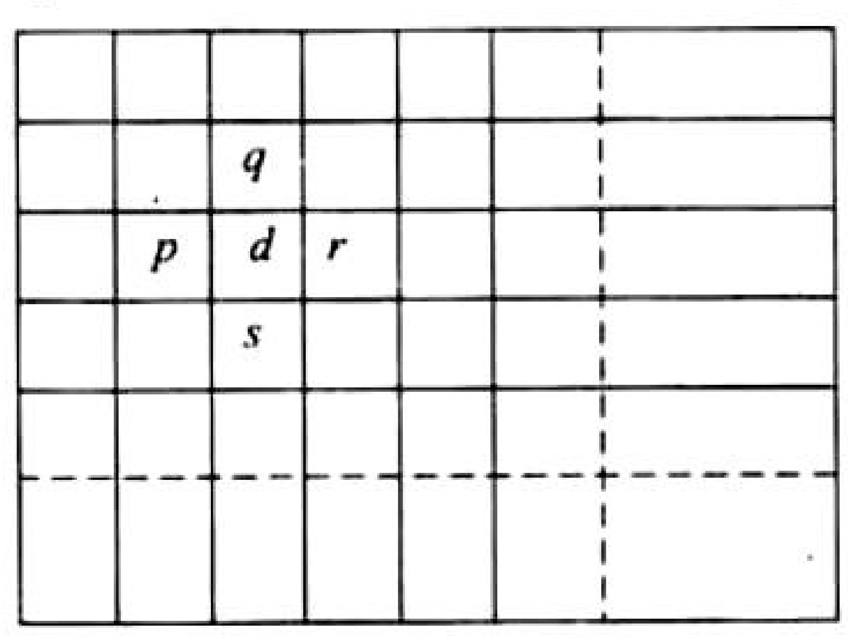
$\Rightarrow$ $n - 1 = 3m - 3$ or $3m = n + 2$

$\Rightarrow$ $\frac{m}{1} = \frac{n+2}{3} = k$ say

$\therefore$ $m = k, n = 3k - 2$

By giving k different values, we get integral values of m and n. Hence, there can be infinite number of G.P.s whose any three terms will be 8, 12, 27 (not consecutive).

5. For any square there can be at most four neighboring squares.



Let for a square having largest number d, p, q, r, s be written.
Then according to the question, $p + q + r + s = 4d$

$\Rightarrow \quad (d - p) + (d - q) + (d - r) + (d - s) = 0$

Sum of four positive numbers can be zero only if these are zeros individually.

Therefore, $d - p = d - q = d - r = d - s = 0$

$\Rightarrow \quad p = q = r = s = d$

Hence, all the numbers written are same.

6. We have $a + b + c = 25$ (1)

$2a = b + 2$ (2)

and $c^2 = 18b$ (3)

Eliminating 'a' from (1) and (2)

$$b = 16 - \frac{2c}{3}$$

Then from (3), $c^2 = 18\left(16 - \frac{2c}{3}\right)$

$\Rightarrow \quad c^2 + 12c - 18 \times 16 = 0$

$\Rightarrow \quad (c - 12)(c + 24) = 0$

$c = -24$ is not possible since it does not lie between 2 and 18.

Hence $c = 12$.

Then from (3), $b = 8$ and finally from (2), $a = 5$.

7. Let the three distinct real numbers be α/r, α, αr. As the sum of squares of three numbers is S^2, we have

$$\frac{\alpha^2}{r^2} + \alpha^2 + \alpha^2 r^2 = S^2$$

or $$\frac{\alpha^2 (1 + r^2 + r^4)}{r^2} = S^2 \quad (1)$$

Sum of numbers is aS. Hence,

$$\frac{\alpha}{r} + \alpha + \alpha r = aS$$

or $$\frac{\alpha(1 + r + r^2)}{r} = aS \quad (2)$$

Dividing Eq. (1) by the square of Eq. (2), we get

$$\frac{\alpha^2 (1 + r^2 + r^4)}{r^2} \times \frac{r^2}{\alpha^2 (1 + r + r^2)^2} = \frac{S^2}{a^2 S^2}$$

or $$\frac{(1 + 2r^2 + r^4) - r^2}{(1 + r + r^2)^2} = \frac{1}{a^2}$$

or $$\frac{(1 + r + r^2)(1 - r + r^2)}{(1 + r + r^2)^2} = \frac{1}{a^2}$$

or $\quad a^2 r^2 - a^2 r + a^2 = 1 + r + r^2$

or $\quad (a^2 - 1) r^2 - (a^2 + 1) r + (a^2 - 1) = 0$

or $$r^2 + \left(\frac{1 + a^2}{1 - a^2}\right) r + 1 = 0 \quad (3)$$

For real values of r,

$D \geq 0$

$$\Rightarrow \quad \left(\frac{1 + a^2}{1 - a^2}\right)^2 - 4 \geq 0$$

or $\quad 1 + 2a^2 + a^4 - 4 + 8a^2 - 4a^4 \geq 0$

or $\quad 3a^4 - 10a^2 + 3 \leq 0$

or $\quad (3a^2 - 1)(a^2 - 3) \leq 0$

or $$\left(a^2 - \frac{1}{3}\right)(a^2 - 3) \leq 0$$

Clearly, the above inequality holds for $1/3 \leq a^2 \leq 3$.

But from Eq. (3), $a \neq 1$.

$$\therefore \quad a^2 \in \left(\frac{1}{3}, 1\right) \cup (1, 3)$$

8. Given that $\log_3 2$, $\log_3 (2^x - 5)$, $\log_3 (2^x - 7/2)$ are in A.P. Hence,

$$2 \log_3 (2^x - 5) = \log_3 \left(2^x - \frac{7}{2}\right) + \log_3 2$$

or $$(2^x - 5)^2 = 2\left(2^x - \frac{7}{2}\right)$$

or $\quad (2^x)^2 - 10 \times 2^x + 25 - 2 \times 2^x + 7 = 0$

or $\quad (2^x)^2 - 12 \times 2^x + 32 = 0$

Let $2^x = y$. Then we get

$y^2 - 12y + 32 = 0$

or $\quad (y - 4)(y - 8) = 0$

$\Rightarrow \quad y = 4$ or 8

$\Rightarrow \quad 2^x = 2^2$ or 2^3

$\Rightarrow \quad x = 2$ or 3

But for $\log_3(2^x - 5)$ and $\log_3 (2^x - 7/2)$ to be defined,

$$2^x - 5 > 0 \text{ and } 2^x - \frac{7}{2} > 0$$

$$\Rightarrow \quad 2^x > 5 \text{ and } 2^x > \frac{7}{2}$$

$\Rightarrow \quad 2^x > 5$

$\Rightarrow \quad x \neq 2$

and therefore, $x = 3$.

9. Let a and b be two numbers and $A_1, A_2, A_3, \ldots, A_n$ be n A.M.'s between a and b. Then, $a, A_1, A_2, \ldots, A_n, b$ are in A.P. There are $n + 2$ terms in the series. Now,

$$a + (n + 1) d = b \text{ or } d = \frac{b - a}{n + 1}$$

$$\Rightarrow \quad A_1 = p = a + \frac{b - a}{n + 1} = \frac{an + b}{n + 1} \quad (1)$$

The first H.M. between a and b, when n H.M.'s are inserted between a and b can be obtained by replacing a by $1/a$ and b by $1/b$ in Eq. (1) and then taking its reciprocal. Therefore,

$$q = \frac{1}{\dfrac{\left(\frac{1}{a}\right) n + \frac{1}{b}}{n + 1}} = \frac{(n + 1) ab}{bn + a} \quad (2)$$

Substituting $b = p(n + 1) - an$ [from (1)] in Eq. (2), we get

$aq + nq [p(n + 1) - an] = (n + 1) a [p(n + 1) - an]$

or $\quad a^2 n(n + 1) + a[q(1 - n^2) - p(n + 1)^2] + npq(n + 1) = 0$

or $\quad na^2 - [(n + 1) p + (n - 1) q] a + npq = 0$

$\Rightarrow \quad D \geq 0$ ($\because a$ is real)

$\Rightarrow \quad [(n + 1) p + (n - 1) q]^2 - 4n^2 pq \geq 0$

or $\quad (n - 1)^2 q^2 + \{2(n^2 - 1) - 4n^2\} pq + (n + 1)^2 p^2 \geq 0$

or $$q^2 - 2\frac{n^2+1}{(n-1)^2}pq + \left(\frac{n+1}{n-1}\right)^2 p^2 \ge 0$$

or $$\left[q - p\left(\frac{n+1}{n-1}\right)^2\right][q-p] \ge 0$$

[On factorizing by discriminent method]

Hence, q cannot lie between p and $p\left(\frac{n+1}{n-1}\right)^2$.

10. According to the question, we have

$$S_1 = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots \infty = \frac{1}{1-\frac{1}{2}} = 2$$

$$S_2 = 2 + 2\times\frac{1}{3} + 2\times\left(\frac{1}{3}\right)^2 + \cdots \infty = \frac{2}{1-\frac{1}{3}} = 3$$

$$S_3 = 3 + 3\times\frac{1}{4} + 3\times\left(\frac{1}{4}\right)^2 + \cdots \infty = \frac{3}{1-\frac{1}{4}} = 4$$

$$S_n = n + n\times\frac{1}{n+1} + n\times\left(\frac{1}{n+1}\right)^2 + \cdots \infty$$

$$= \frac{n}{1-\frac{1}{n+1}} = (n+1)$$

$$\therefore \quad S_1^2 + S_2^2 + S_3^2 + \cdots + S_{2n-1}^2$$

$$= 2^2 + 3^2 + 4^2 + \cdots + (n+1)^2 + \cdots + (2n)^2$$

$$= \left(\sum_{r=1}^{2n} r^2\right) - 1$$

$$= \frac{2n(2n+1)(4n+1)}{6} - 1^2$$

$$= \frac{n(2n+1)(4n+1) - 3}{3}$$

11. Let a and b be the two numbers and let H be the harmonic mean between them. Then, $H = 4$ (given). Since A, G, H are in G.P., we have,

$$G^2 = AH$$
$$= 4A$$

But

$$2A + G^2 = 27 \qquad \text{(given)}$$

$$\therefore \quad 6A = 27 \qquad [\because G^2 = 4A]$$

or $$A = \frac{9}{2}$$

$$\Rightarrow \quad \frac{a+b}{2} = \frac{9}{2}$$

or $a + b = 9$

Now, $G^2 = 4A$ and $A = 9/2 \Rightarrow G^2 = 18 \Rightarrow ab = 18$. The quadratic equation having a, b as its roots is

$$x^2 - (a+b)x + ab = 0$$

$$\Rightarrow \quad x^2 - 9x + 18 = 0 \qquad [\because a + b = 9 \text{ and } ab = 18]$$

$$\Rightarrow \quad x = 3, 6$$

Hence, the two numbers are 3 and 6.

12. Solving the system of equations, $u + 2v + 3w = 6$, $4u + 5v + 6w = 12$, and $6u + 9v = 4$, we get

$$u = -1/3,\ v = 2/3,\ w = 5/3$$

$$\therefore \quad u + v + w = 2 \text{ and } \frac{1}{u} + \frac{1}{v} + \frac{1}{w} = -\frac{9}{10}$$

Let r be the common ratio of the G.P. a, b, c, d. Then $b = ar$, $c = ar^2$, $d = ar^3$. Then the first equation

$$\left(\frac{1}{u} + \frac{1}{v} + \frac{1}{w}\right)x^2 + [(b-c)^2 + (c-a)^2 + (d-b)^2]x + (u+v+w) = 0 \text{ becomes}$$

$$-\frac{9}{10}x^2 + [(ar - ar^2)^2 + (ar^2 - a)^2 + (ar^3 - ar)^2]x + 2 = 0$$

or $$9x^2 - 10a^2(1-r)^2(r^4 + 2r^3 + 3r^2 + 2r + 1)x - 20 = 0$$

or $$9x^2 - 10a^2(1-r)^2(1 + r + r^2)^2 x - 20 = 0$$

or $$9x^2 - 10a^2(1 - r^3)^2 x - 20 = 0 \qquad (1)$$

The second equation is

$$20x^2 + 10(a - ar^3)^2 x - 9 = 0$$

or $$20x^2 + 10a^2(1 - r^3)^2 x - 9 = 0 \qquad (2)$$

Since (2) can be obtained by changing x to $1/x$, so Eqs. (1) and (2) have reciprocal roots.

13. Let $a - 3d$, $a - d$, $a + d$, and $a + 3d$ be any consecutive terms of an A.P. with common difference $2d$. Hence,

$$P = (2d)^4 + (a-3d)(a-d)(a+d)(a+3d)$$
$$= 16d^4 + (a^2 - 9d^2)(a^2 - d^2)$$
$$= (a^2 - 5d^2)^2$$

which is an integer.

14. Clearly, $A_1 + A_2 = a + b$. Now,

$$\frac{1}{H_1} + \frac{1}{H_2} = \frac{1}{a} + \frac{1}{b}$$

or $$\frac{H_1 + H_2}{H_1 H_2} = \frac{a+b}{ab} = \frac{A_1 + A_2}{G_1 G_2}$$

or $$\frac{G_1 G_2}{H_1 H_2} = \frac{A_1 + A_2}{H_1 + H_2}$$

Also,

$$\frac{1}{H_1} = \frac{1}{a} + \frac{1}{3}\left(\frac{1}{b} - \frac{1}{a}\right) \text{ or } H_1 = \frac{3ab}{2b + a}$$

and

$$\frac{1}{H_2} = \frac{1}{a} + \frac{2}{3}\left(\frac{1}{b} - \frac{1}{a}\right) \text{ or } H_2 = \frac{3ab}{2a + b}$$

$$\Rightarrow \quad \frac{A_1 + A_2}{H_1 + H_2} = \frac{a + b}{3ab\left(\frac{1}{2b+a} + \frac{1}{2a+b}\right)}$$

$$= \frac{(2b+a)(2a+b)}{9ab}$$

15. Given that a, b, and c are in A.P. Hence,

$$2b = a + c \qquad (1)$$

a^2, b^2, c^2 are in H.P. Hence,

$$\frac{1}{b^2} - \frac{1}{a^2} = \frac{1}{c^2} - \frac{1}{b^2}$$

or $$\frac{(a-b)(a+b)}{b^2 a^2} = \frac{(b-c)(b+c)}{b^2 c^2}$$

or $\quad ac^2 + bc^2 = a^2b + a^2c \quad [\because a - b = b - c]$

or $\quad ac(c-a) + b(c-a)(c+a) = 0$

or $\quad (c-a)(ab+bc+ca) = 0$

$\Rightarrow \quad c - a = 0$ or $ab + bc + ca = 0$

For $c = a$, from (1), $a = b = c$. For $(a+c)b + ca = 0$, from (1),

$$2b^2 + ca = 0$$

or $\quad b^2 = a\left(\frac{-c}{2}\right)$

Hence, $a, b, -c/2$ are in G.P.

16. Given that a, b, c are positive real numbers. To prove that $(a+1)^7 (b+1)^7 (c+1)^7 > 7^7 a^4 b^4 c^4$.

$$\text{L.H.S.} = (1+a)^7 (1+b)^7 (1+c)^7$$
$$= [(1+a)(1+b)(1+c)]^7$$
$$= [1 + a + b + c + ab + bc + ca + abc]^7 \quad (1)$$

Now,

A.M. $\geq$ G.M.

$$\Rightarrow \frac{a+b+c+ab+bc+ca+abc}{7} \geq \left(a^4b^4c^4\right)^{1/7}$$

or $\quad (a+b+c+ab+bc+ca+abc)^7 \geq 7^7 (a^4 b^4 c^4) \quad (2)$

From Eqs. (1) and (2), we get

$$[(1+a)(1+b)(1+c)]^7 > 7^7 a^4b^4c^4$$

17. $G_k = (a_1a_2 \cdots a_k)^{1/k}$

$$= a_1 \left(r^{1+2+\cdots+(k-1)}\right)^{1/k}$$
$$= a_1 r^{\frac{k-1}{2}} \quad (1)$$

$$A_k = \frac{a_1 + a_2 + \cdots + a_k}{k}$$
$$= \frac{a_1(1 + r + \cdots + r^{k-1})}{k}$$
$$= \frac{a_1(r^k - 1)}{(r-1) \times k} \quad (2)$$

$$H_k = \frac{k}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}}$$
$$= \frac{a_1 k}{\left(1 + \frac{1}{r} + \cdots + \frac{1}{r^{k-1}}\right)}$$
$$= \frac{a_1 k (r-1) r^{k-1}}{r^k - 1} \quad (3)$$

From (1), (2), and (3), we get

$$G_k = (A_k H_k)^{1/2}$$

$$\Rightarrow \prod_{k=1}^{n} G_k = \prod_{k=1}^{n} (A_k H_k)^{1/2}$$

or $\quad \left(\prod_{k=1}^{n} G_k\right)^{1/n} = (A_1 A_2 \cdots A_n \times H_1 H_2 \cdots H_n)^{1/2n}$

18. Total number of runs scored

$$= \sum_{k=1}^{n} k \cdot 2^{n-k+1}$$
$$= 2^{n+1} \sum_{k=1}^{n} \frac{k}{2^k}$$
$$= 2^{n+1}\left[1\left(\frac{1}{2}\right) + 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + \cdots + (n-1)\left(\frac{1}{2}\right)^{n-1} + n\left(\frac{1}{2}\right)^n\right]$$

$$S = 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + \cdots + (n-1)\left(\frac{1}{2}\right)^{n-1} + n\left(\frac{1}{2}\right)^n$$

$$\therefore \quad \frac{1}{2}S = 1\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^4 + \cdots + (n-1)\left(\frac{1}{2}\right)^n + n\left(\frac{1}{2}\right)^{n+1}$$

Subtracting

$$\frac{1}{2}S = \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \cdots + \left(\frac{1}{2}\right)^n - n\left(\frac{1}{2}\right)^{n+1}$$

$$\Rightarrow \frac{1}{2}S = \frac{\frac{1}{2}\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}} - n\left(\frac{1}{2}\right)^{n+1}$$

$$\Rightarrow S = 2\left(1 - \frac{1}{2^n}\right) - n\left(\frac{1}{2^n}\right)$$

$\therefore$ Total number of runs scored

$= 2[2^{n+1} - n - 2] \quad$ (sum of A.G.P)

According to question

$$\left(\frac{n+1}{4}\right)(2^{n+1} - n - 2) = 2[2^{n+1} - n - 2]$$

$$\Rightarrow \left(\frac{n+1}{4}\right) = 2$$

$$\Rightarrow n = 7$$

19. $a_n = \frac{3}{4} - \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^3 + \cdots + (-1)^{n-1}\left(\frac{3}{4}\right)^n$

$$= \frac{\frac{3}{4}\left(1 - \left(-\frac{3}{4}\right)^n\right)}{1 - \left(-\frac{3}{4}\right)}$$

$$= \frac{3}{7}\left(1 - \left(-\frac{3}{4}\right)^n\right)$$

Now, $b_n = 1 - a_n$ and $b_n > a_n$ for $n \geq n_0$.

$\therefore \quad 1 - a_n > a_n$ or $2a_n < 1$

or $\quad \frac{6}{7}\left[1 - \left(-\frac{3}{4}\right)^n\right] < 1$

or $\quad -\left(-\frac{3}{4}\right)^n < \frac{1}{6}$

or $\quad (-3)^{n+1} < 2^{2n-1}$

For n to be even, inequality always holds. For n to be odd, it holds for $n \geq 7$. Therefore, the least natural number for which it holds is 6.